Int. J. Solids Structures, 1971, Vol. 7, pp. 789 to 803. Pergamon Press. Printed in Great Britain

FREE VIBRATIONS OF BEAM-LIKE STRUCTURES

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Abstract—The purpose of this paper is to present a generalized beam theory by which a number of beam-like structures may advantageously be treated as beams.

The equations of motion are deduced including the effect of rotatory inertia. Considering free harmonic vibrations, the resulting eigenvalue problem, which consists of two simultaneous second order differential equations and four boundary conditions, is solved by a conventional technique using successive approximations.

The power of this theory is demonstrated in a few simple examples. The natural frequencies of some beam-like trusses are computed and the results are compared with results of the Bernoulli–Euler and Timoshenko beam theories.

1. INTRODUCTION

TRUSSES, even slender ones, cannot in general be treated adequately as one-dimensional structures by the existing beam theories. However, the possibility of applying a one-dimensional beam theory to the calculation of beam-like structures would be a considerable advantage due to the simplification. It therefore seems motivated to look for a theory more general than the Bernoulli–Euler or the Timoshenko beam theories, by which a larger class of elastic structures could be appropriately handled as "beams". In this paper such a theory is developed, and a numerical procedure for the calculation of natural frequencies is introduced. A few examples are included to show how satisfactorily such structures may be treated as generalized beams even when conventional beam theories prove to be inadequate.

In the derivation of the differential equation for vibrating beams Bresse [1] was the first to include the effect of rotatory inertia. Timoshenko [2] included the effect of the transverse shear force in addition to this. He replaced Bernoulli's assumption that plane cross-sections stay perpendicular to the beam by the assumption that the angle between the middle line of the beam and the normal of the cross-section is proportional to the transverse shear force. This leads to a system of two differential equations, which has been solved by a number of authors. For example, Anderson [3] gives a series-solution, and Huang [4] applies the methods of Ritz and Galerkin. Herrmann [5] has given formal solutions for forced vibrations with time dependent boundary conditions in uniform Timoshenko beams.

2. A GENERALIZED BEAM THEORY

The beam in consideration is straight, elastic and symmetrical about the plane in which all loads act. In this section we shall derive constitutive equations and equations of equilibrium for this beam.

In Fig. 1 the positive directions are defined for forces and couples acting on a small element of the beam. The bending moment is denoted M, the transverse shear force is

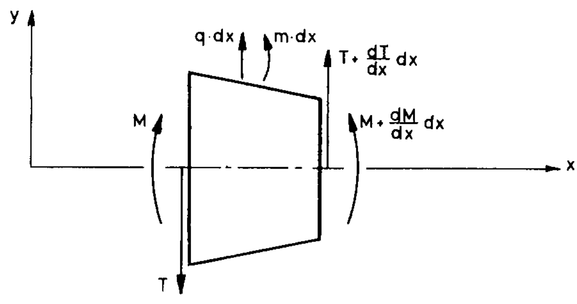


FIG. 1. Loads on an element of the beam.

denoted T, an external force is $q\,.\,\mathrm{d}x$ and an external couple is $m\,.\,\mathrm{d}x$. Figure 2 shows the positive directions of the transverse deflection y, the angle of rotation of the cross-section v and the angle γ between the normal of the cross-section and the middle line.

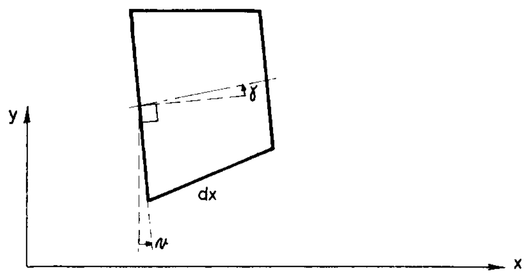


FIG. 2. Deformations of an element of the beam.

Constitutive equations

In order to derive the constitutive equations we must find the components of the generalized strain vector corresponding to the generalized stress components M and T. If the beam material is linearly elastic, the strain energy contained in a small element of the beam is

$$\mathrm{d}W_i = \tfrac{1}{2}M\,\mathrm{d}v + \tfrac{1}{2}T\gamma\,\mathrm{d}x = \frac{1}{2}\left\{M\frac{\mathrm{d}v}{\mathrm{d}x} + T\left(\frac{\mathrm{d}y}{\mathrm{d}x} - v\right)\right\}\mathrm{d}x. \tag{2.1}$$

Consequently, the components of the generalized strain vector are $\mathrm{d}v/\mathrm{d}x$ and $\mathrm{d}y/\mathrm{d}x - v$. Thus we can write the general linear constitutive equations in the form

$$\frac{\mathrm{d}v}{\mathrm{d}x} = a_{11}M + a_{12}T \tag{2.2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - v = a_{21}M + a_{22}T \tag{2.3}$$

where $a_{11}(x)$, $a_{12}(x)$, $a_{21}(x)$ and $a_{22}(x)$ are characteristic functions of the beam.

In the special case $a_{12} \equiv a_{21} \equiv 0$, $a_{11} = 1/EI$ and $a_{22} = \mu/GA$, equations (2.2) and (2.3) reduce to the constitutive equations of Timoshenko beams, and if, furthermore, $a_{22} \equiv 0$, these equations reduce to the Bernoulli–Euler beam equation.

In the last section of this paper the beam functions are determined for a special example of a beam-like structure, and the results of the present beam theory are compared with results of the Bernoulli–Euler and Timoshenko theories.

Equations of equilibrium

For a small element of the beam, equilibrium of the forces results in the equation

$$-\frac{\mathrm{d}T}{\mathrm{d}x} = q \tag{2.4}$$

and equilibrium of the couples gives

$$-\frac{\mathrm{d}M}{\mathrm{d}x} - T = m. \tag{2.5}$$

Using equations (2.2) and (2.3) and introducing the expression

$$D(x) = a_{11}(x)a_{22}(x) - a_{12}(x)a_{21}(x) \tag{2.6}$$

the bending moment M and the shear force T may be expressed in the following way

$$M = \frac{a_{22}}{D}\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{a_{12}}{D}\left(\frac{\mathrm{d}y}{\mathrm{d}x} - v\right) \tag{2.7}$$

$$T = -\frac{a_{21}}{D}\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{a_{11}}{D}\left(\frac{\mathrm{d}y}{\mathrm{d}x} - v\right). \tag{2.8}$$

We substitute the bending moment M and the shear force T from these equations in the equations of equilibrium (2.4) and (2.5). This leads to the following system of differential equations

$$\left\{\frac{a_{21}}{D}v' - \frac{a_{11}}{D}(y' - v)\right\}' = q \tag{2.9}$$

$$\left\{-\frac{a_{22}}{D}v' + \frac{a_{12}}{D}(y' - v)\right\}' + \left\{\frac{a_{21}}{D}v' - \frac{a_{11}}{D}(y' - v)\right\} = m \tag{2.10}$$

where the symbol $'$ denotes the derivative d/dx.

In the following we shall often write the equations of equilibrium (2.9) and (2.10) in the form

$$\bar{L}[\bar{z}] = \bar{r} \tag{2.11}$$

where the load vector $\bar{r} = \begin{Bmatrix} q \\ m \end{Bmatrix}$ is a column vector, and $\bar{L}[\ ]$ stands for a column vector in which the components are linear differential operators. The vector $\bar{z} = (y, v)$ is a row vector.

Boundary conditions

The system of differential equations (2.9) and (2.10) can be reduced to a system of four linear ordinary differential equations of first order if we introduce two new variables, $u = y'$ and $w = v'$. Consequently we need exactly four linearly independent boundary conditions to obtain a unique solution of the equations of equilibrium [6].

Some typical cases of the two boundary conditions that appear at each end of the beam are shown in Table 1.

TABLE 1

Clamped	$y = v = 0$
Simply supported	$y = M = 0$
Free	$T = M = 0$
Elastically supported and elastically clamped	$T = S_1 y, \quad M = S_2 v$

If we replace the bending moment M and the shear force T by the expressions (2.7) and (2.8), all boundary conditions of Table 1 become linear homogeneous differential expressions in y and v. Consequently, we can write the set of boundary conditions in the homogeneous form

$$U_n[\bar{z}] = 0, \qquad \text{for} \quad n = 1, 2, 3, 4. \tag{2.12}$$

Maxwell's theorem

We consider two conservative load vectors $\bar{r}_1(x)$ and $\bar{r}_2(x)$ for the beam. The corresponding solutions of the boundary value problem defined by equations (2.11) and (2.12) are $\bar{z}_1$ and $\bar{z}_2$. Then Maxwell's theorem for the beam can be written in the form

$$\int_0^l \bar{z}_1 \bar{r}_2 \, dx = \int_0^l \bar{z}_2 \bar{r}_1 \, dx \tag{2.13}$$

or in the form

$$\int_0^l \{\bar{z}_1 \bar{L}[\bar{z}_2] - \bar{z}_2 \bar{L}[\bar{z}_1]\} \, dx = 0 \tag{2.14}$$

where l is the length of the beam.

After substitution of equations (2.9) and (2.10) in equation (2.14), we can prove that a necessary condition for Maxwell's theorem is

$$a_{12} \equiv a_{21}. \tag{2.15}$$

We note the analogy between the symmetrical property (2.15) of the coefficient functions for a continuous system and the corresponding property of the flexibility matrix for a system with a finite number of degrees of freedom.

Strain energy

Integrating the strain energy (2.1) of a small beam element over the length of the beam, we find the total strain energy

$$W_i = \frac{1}{2} \int_0^l (a_{11} M^2 + (a_{12} + a_{21}) M T + a_{22} T^2) \, dx. \tag{2.16}$$

As the strain energy has to be positive, the integrand of equation (2.16) can never be negative for any combination of the bending moment M and the transverse shear force T. This leads to the following necessary conditions

$$a_{11} \geq 0, \qquad a_{22} \geq 0. \tag{2.17}$$

If either $a_{11} = 0$ or $a_{22} = 0$ equations (2.16) and (2.15) result in the additional condition

$$a_{12} = a_{21} = 0. \tag{2.18}$$

However, if $a_{11} > 0$ and $a_{22} > 0$, which is the interesting case in this paper, we find instead the following condition

$$-2\surd(a_{11}a_{22}) < a_{12}+a_{21} < 2\surd(a_{11}a_{22}). \tag{2.19}$$

By application of equation (2.15) this condition can be written in the form

$$D = a_{11}a_{22}-a_{12}a_{21} > 0. \tag{2.20}$$

Consequently, the denominator appearing in equations (2.7) and (2.8) is always positive.

3. VIBRATIONS OF THE BEAM

In the following we deduce the equations of motion of the beam. This leads to an eigenvalue problem, for which we shall introduce the orthogonality relation and Rayleigh's quotient.

Free vibrations

The external loads are equal to the d'Alembert loads $q(x, t) = -\rho A(\partial^2 Y/\partial t^2)$ and $m(x, t) = -\rho I(\partial^2 V/\partial t^2)$, where the cross-sectional area is $A(x)$, the area moment of inertia of the cross-section is $I(x)$, the mass density of the beam material is ρ and t denotes the time. Substituting these loads in equations (2.9) and (2.10) we obtain the equations of motion

$$\frac{\partial}{\partial x}\left\{\frac{a_{21}}{D}\frac{\partial V}{\partial x}-\frac{a_{11}}{D}\left(\frac{\partial Y}{\partial x}-V\right)\right\} = -\rho A\frac{\partial^2 Y}{\partial t^2} \tag{3.1}$$

$$\frac{\partial}{\partial x}\left\{-\frac{a_{22}}{D}\frac{\partial V}{\partial x}+\frac{a_{12}}{D}\left(\frac{\partial Y}{\partial x}-V\right)\right\}+\left\{\frac{a_{21}}{D}\frac{\partial V}{\partial x}-\frac{a_{11}}{D}\left(\frac{\partial Y}{\partial x}-V\right)\right\} = -\rho I\frac{\partial^2 V}{\partial t^2}. \tag{3.2}$$

We consider solutions in the form

$$\begin{aligned} Y(x, t) &= y(x)\,\mathrm{e}^{ipt} \\ V(x, t) &= v(x)\,\mathrm{e}^{ipt}. \end{aligned} \tag{3.3}$$

Replacing Y and V in equations (3.1) and (3.2) by (3.3) and taking $\lambda = p^2$, we find the following two simultaneous, linear, homogeneous, ordinary differential equations

$$\left\{\frac{a_{21}}{D}v'-\frac{a_{11}}{D}(y'-v)\right\}' = \lambda\rho A y \tag{3.4}$$

$$\left\{-\frac{a_{22}}{D}v'+\frac{a_{12}}{D}(y'-v)\right\}'+\left\{\frac{a_{21}}{D}v'-\frac{a_{11}}{D}(y'-v)\right\} = \lambda\rho I v. \tag{3.5}$$

Together with four boundary conditions, equations (3.4) and (3.5) form an eigenvalue problem. Making use of equations (2.11) and (2.12), and introducing the vector operator $\bar{N}[\bar{z}] = \begin{Bmatrix} \rho A y \\ \rho I v \end{Bmatrix}$ we can write the eigenvalue problem in the form

$$\begin{aligned} \bar{L}[\bar{z}] &= \lambda \bar{N}[\bar{z}] \\ U_n[\bar{z}] &= 0, \qquad \text{for} \quad n = 1, 2, 3, 4. \end{aligned} \tag{3.6}$$

We now define the following two classes of vectors:

1. A comparison vector $\bar{\mathbf{z}} = (\mathbf{y}, \mathbf{v})$ is any pair of functions that satisfies all the boundary conditions and possesses continuous derivatives of second order.
2. An eigenvector $\bar{z} = (y, v)$ is any pair of functions that solves the eigenvalue problem.

The vector $\bar{z} \equiv \bar{0}$ is excluded from both classes.

The orthogonality relation

An eigenvalue problem in the form (3.6) is called self-adjoint when it satisfies the following two relations

$$\int_0^l \{\bar{\mathbf{z}}_1 \bar{L}[\bar{\mathbf{z}}_2] - \bar{\mathbf{z}}_2 \bar{L}[\bar{\mathbf{z}}_1]\} \, dx = 0 \tag{3.7}$$

$$\int_0^l \{\bar{\mathbf{z}}_1 \bar{N}[\bar{\mathbf{z}}_2] - \bar{\mathbf{z}}_2 \bar{N}[\bar{\mathbf{z}}_1]\} \, dx = 0 \tag{3.8}$$

for any two comparison vectors $\bar{\mathbf{z}}_1$ and $\bar{\mathbf{z}}_2$. By application of equations (3.4) and (3.5) it can be shown that equation (3.7) is satisfied for any combination of the boundary conditions of Table 1 and that equation (3.8) is satisfied regardless of the boundary conditions.

Since the eigenvectors $\bar{z}_i$ are included in the class of comparison vectors, we find, by substitution of (3.6) in (3.7),

$$\int_0^l \{\bar{z}_i \bar{N}[\bar{z}_j]\lambda_j - \bar{z}_j \bar{N}[\bar{z}_i]\lambda_i\} \, dx = 0. \tag{3.9}$$

Making use of equation (3.8), we convert (3.9) into the equation

$$(\lambda_i - \lambda_j) \int_0^l \bar{z}_i \bar{N}[\bar{z}_j] \, dx = 0 \tag{3.10}$$

which leads to the orthogonality relation for the beam

$$\int_0^l (\rho A y_i y_j + \rho I v_i v_j) \, dx = 0, \qquad \text{for} \quad \lambda_i \neq \lambda_j. \tag{3.11}$$

Rayleigh's quotient

An eigenvalue problem in the form (3.6) is called full-definite if it satisfies the following two relations

$$\int_0^l \bar{\mathbf{z}} \bar{L}[\bar{\mathbf{z}}] \, dx > 0 \tag{3.12}$$

$$\int_0^l \bar{\mathbf{z}} \bar{N}[\bar{\mathbf{z}}] \, dx > 0 \tag{3.13}$$

for any comparison vector $\bar{\mathbf{z}}$. Application of equations (3.4) and (3.5) shows that (3.13) is satisfied regardless of the boundary conditions and that (3.12) is satisfied for any combination of the boundary conditions of Table 1, except when the beam is free at both ends.

Now we introduce Rayleigh's quotient

$$R[\bar{\mathbf{z}}] = \frac{\int_0^l \bar{\mathbf{z}}\bar{L}[\bar{\mathbf{z}}]\,\mathrm{d}x}{\int_0^l \bar{\mathbf{z}}\bar{N}[\bar{\mathbf{z}}]\,\mathrm{d}x}. \tag{3.14}$$

When the boundary conditions are a combination of the conditions shown in Table 1, Rayleigh's quotient can be written in the form

$$R[\mathbf{y},\mathbf{v}] = \frac{\displaystyle\int_0^l \left\{\frac{a_{22}}{D}(\mathbf{v}')^2 - \left(\frac{a_{12}}{D}+\frac{a_{21}}{D}\right)\mathbf{v}'(\mathbf{y}'-\mathbf{v}) + \frac{a_{11}}{D}(\mathbf{y}'-\mathbf{v})^2\right\}\mathrm{d}x}{\int_0^l (\rho A\mathbf{y}^2 + \rho I\mathbf{v}^2)\,\mathrm{d}x}. \tag{3.15}$$

Using Green's functions we can go through a number of proofs analogous to those of Collatz [7] but extended to cover also eigenvalue problems of the type (3.6). Assuming that the eigenvalue problem is self-adjoint and full-definite, we can prove that if λ_1 is the smallest eigenvalue and $\bar{\mathbf{z}}$ is any comparison vector, Rayleigh's quotient satisfies the inequality

$$R[\bar{\mathbf{z}}] \geq \lambda_1. \tag{3.16}$$

If $\bar{\mathbf{z}}$ is restricted to being any comparison vector that is orthogonal to the first $s-1$ eigenvectors, we find

$$R[\bar{\mathbf{z}}] \geq \lambda_s. \tag{3.17}$$

4. NUMERICAL SOLUTION

The eigenvalue problem (3.6) can be solved by successive approximations. We make use of the iteration scheme

$$\begin{aligned} \bar{L}[h_i, g_i] &= \bar{N}[h_{i-1}, g_{i-1}] \\ & \qquad\qquad \text{for} \quad i = 1, 2, \ldots \\ U_n[h_i, g_i] &= 0 \end{aligned} \tag{4.1}$$

which can be started with any pair of integrable functions $(h_0, g_0) \not\equiv (0, 0)$. Depending on the size of λ_1, the numerical values of $h_i(x)$ and $g_i(x)$ will tend to converge against either 0 or ∞, and consequently, it is practical to normalize the comparison vector (h_i, g_i) after each iteration:

$$(H_i, G_i) = \frac{(h_i, g_i)}{\sqrt{[\int_0^l (\rho A h_i^2 + \rho I g_i^2)\,\mathrm{d}x]}}. \tag{4.2}$$

The iteration is stopped when the vector (H_i, G_i) becomes stationary or, in practice, when

$$\int_0^l \{\rho A(H_i - H_{i-1})^2 + \rho I(G_i - G_{i-1})^2\}\,\mathrm{d}x < \varepsilon \tag{4.3}$$

where ε is a small positive constant. The smallest eigenvalue is determined approximately by the Rayleigh quotient

$$\lambda_1 \simeq R[H_i, G_i]. \tag{4.4}$$

By analogy with Koch's method [7] we can determine the sth eigenvector. We make use of the iteration scheme

$$\left.\begin{aligned} \bar{L}[h_i^*, g_i^*] &= \bar{N}[h_{i-1}, g_{i-1}] \\ U_\mu[h_i^*, g_i^*] &= 0 \end{aligned}\right\} \quad \text{for} \quad i = 1, 2, \ldots \tag{4.5}$$

together with the orthogonalization

$$(h_i, g_i) = (h_i^*, g_i^*) - \sum_{j=1}^{s-1} (y_j, v_j) \frac{\int_0^l (h_i^*, g_i^*) \bar{N}[y_j, v_j]\,\mathrm{d}x}{\int_0^l (y_j, v_j) \bar{N}[y_j, v_j]\,\mathrm{d}x} \tag{4.6}$$

in which the first $s-1$ eigenvectors (y_j, v_j) are approximated by those determined by successive iterations. Using equation (4.2), we further normalize after each iteration, and finally, according to (3.17), the sth eigenvalue is

$$\lambda_s \simeq R[H_i, G_i]. \tag{4.7}$$

Formal integration of the differential equations

The easiest way to carry out one step of the iteration (4.1) is by numerical integration. Here we shall deduce the necessary formulas.

Integration of equation (3.4) results in

$$\frac{a_{21}}{D} v' - \frac{a_{11}}{D}(y' - v) = \lambda \left[\int_0^x \rho A y \,\mathrm{d}\xi + C_1 \right] = -T(x) \tag{4.8}$$

and substituting this into equation (3.5), we find by integration

$$-\frac{a_{22}}{D} v' + \frac{a_{12}}{D}(y' - v) = \lambda \left[\int_0^x \left(\rho I v - \int_0^\xi \rho A y \,\mathrm{d}\eta \right) \mathrm{d}\xi - C_1 x + C_2 \right] = -M(x). \tag{4.9}$$

These two equations can be solved with respect to v' and $y' - v$:

$$\begin{Bmatrix} \dfrac{a_{22}}{D} & -\dfrac{a_{12}}{D} \\ -\dfrac{a_{21}}{D} & \dfrac{a_{11}}{D} \end{Bmatrix} \begin{Bmatrix} v' \\ y' - v \end{Bmatrix} = \begin{Bmatrix} M(x) \\ T(x) \end{Bmatrix}$$

$$\begin{Bmatrix} v' \\ y' - v \end{Bmatrix} = \begin{Bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Bmatrix} \begin{Bmatrix} M(x) \\ T(x) \end{Bmatrix}$$

and the results are

$$v' = \lambda\left[-a_{11}\int_0^x\left(\rho I v-\int_0^\xi \rho A y\,\mathrm{d}\eta\right)\mathrm{d}\xi+a_{11}C_1x-a_{11}C_2-a_{12}\int_0^x \rho A y\,\mathrm{d}\xi-a_{12}C_1\right] \quad (4.10)$$

$$y'-v = \lambda\left[-a_{21}\int_0^x\left(\rho I v-\int_0^\xi \rho A y\,\mathrm{d}\eta\right)\mathrm{d}\xi+a_{21}C_1x\right.$$
$$\left.-a_{21}C_2-a_{22}\int_0^x \rho A y\,\mathrm{d}\xi-a_{22}C_1\right]. \quad (4.11)$$

By integration of equation (4.10) we obtain

$$v = \lambda\left[-\int_0^x\left\{a_{11}\int_0^\xi\left(\rho I v-\int_0^\eta \rho A y\,\mathrm{d}\mu\right)\mathrm{d}\eta+a_{12}\int_0^\xi \rho A y\,\mathrm{d}\eta\right\}\mathrm{d}\xi\right.$$
$$\left.-C_1\int_0^x(-a_{11}\xi+a_{12})\,\mathrm{d}\xi-C_2\int_0^x a_{11}\,\mathrm{d}\xi+C_3\right] \quad (4.12)$$

and substituting equation (4.12) in equation (4.11), we find by integration

$$y = \lambda\left[\int_0^x\left\{-\int_0^\xi\left(a_{11}\int_0^\eta\left(\rho I v-\int_0^\mu \rho A y\,\mathrm{d}\kappa\right)\mathrm{d}\mu+a_{12}\int_0^\eta \rho A y\,\mathrm{d}\mu\right)\mathrm{d}\eta-a_{21}\int_0^\xi\left(\rho I v\right.\right.\right.$$
$$\left.\left.-\int_0^\eta \rho A y\,\mathrm{d}\mu\right)\mathrm{d}\eta-a_{22}\int_0^\xi \rho A y\,\mathrm{d}\eta\right\}\mathrm{d}\xi-C_1\int_0^x\left(\int_0^\xi(-a_{11}\eta+a_{12})\,\mathrm{d}\eta-a_{21}\xi+a_{22}\right)\mathrm{d}\xi$$
$$\left.-C_2\int_0^x\left(\int_0^\xi a_{11}\,\mathrm{d}\eta+a_{21}\right)\mathrm{d}\xi+C_3x+C_4\right]. \quad (4.13)$$

Integration constants

During each iteration the four integration constants are determined from the boundary conditions. As an example, substitution of the boundary conditions $y(0) = v(0) = M(l) = T(l) = 0$ in equations (4.8), (4.9), (4.12) and (4.13) leads to the constants

$$\begin{aligned} C_1 &= -\int_0^l \rho A y\,\mathrm{d}x \\ C_2 &= -l\int_0^l \rho A y\,\mathrm{d}x-\int_0^l\left(\rho I v-\int_0^x \rho A y\,\mathrm{d}\xi\right)\mathrm{d}x \\ C_3 &= 0 \\ C_4 &= 0. \end{aligned} \quad (4.14)$$

By analogy with (4.14), the constants have been determined for all combinations of the boundary conditions of Table 1.

The successive approximation method described in this chapter has been used to build up a computer programme that can solve the eigenvalue problem (3.6). In the programme all functions of x are defined only at a finite number of points.

5. APPLICATIONS

It has now been demonstrated how the natural frequencies can be found for a beam with the constitutive equations (2.2) and (2.3), but we still have to determine the beam functions $a_{11}(x)$, $a_{12}(x)$, $a_{21}(x)$ and $a_{22}(x)$ for any specific beam-like structure we wish to examine.

Timoshenko beams

Timoshenko beams are, as already mentioned, a special example of our beam, with the functions $a_{11} = 1/EI$, $a_{22} = \mu/GA$ and $a_{12} \equiv a_{21} \equiv 0$. To find eigenvalues and eigenvectors of Timoshenko beams, the method of successive approximations is a powerful tool, especially in the case of non-uniform beams.

Five mutually orthogonal eigenvectors for a simply supported uniform beam are shown in Fig. 3. The rotations $v(x)$ of the cross-sections are indicated by lines across the

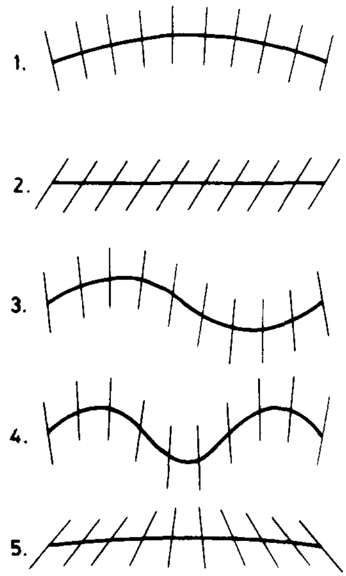


FIG. 3. Eigenvectors for a Timoshenko beam.

deflection curves. By application of the computer programme these five eigenvectors were found to belong to the five smallest eigenvalues of a beam of thin-walled circular cross-section with radius equal to $\frac{1}{4}$ of the length of the beam. If the beam becomes more slender, the eigenvalues belonging to eigenvectors of type 2 and 5 become relatively high [3, 4], while the eigenvectors of type 1, 3 and 4 converge towards the modes of Bernoulli–Euler beams.

Plane trusses

A number of plane trusses can be considered as beam-like structures. Here we shall examine trusses composed of sections of the type shown in Fig. 4.

When the truss in Fig. 4 is loaded at the joints, average values of the beam deflection y and the rotation v can be determined from the joint-displacements, which are found

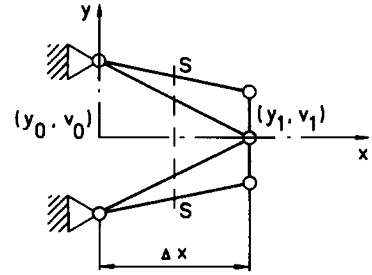


FIG. 4. One section of a plane truss.

by use of the ordinary truss-theory. Now we find y and v for two load systems, which result in linearly independent combinations of the bending moment M and the transverse shear force T in the middle cross-section s–s. Then making use of the following finite difference formulation of equations (2.2) and (2.3):

$$\frac{v_1 - v_0}{\Delta x} = a_{11}M + a_{12}T \tag{5.1}$$

$$\frac{y_1 - y_0}{\Delta x} - \frac{v_1 + v_0}{2} = a_{21}M + a_{22}T \tag{5.2}$$

we are able to determine the four constants a_{11}, a_{12}, a_{21} and a_{22}. If the cross-sections containing the joints are absolutely rigid, the condition (2.15) is satisfied, but since the vertical bars are normally elastic, equation (2.15) is not exactly satisfied in general. Finally, the definition of the deflection y and the rotation v affects the values of the beam functions. In the present calculation only the two joints outside the middle line are used to define y and v.

The beam theory cannot be used to describe the deformations inside one truss-section, but if the truss is composed of more sections, as in Fig. 5, the theory may provide quite a good approximation. Applying the beam theory, we must know the beam functions for all values of x. As a reasonable approximation we consider the functions to be constant within each section of the truss.

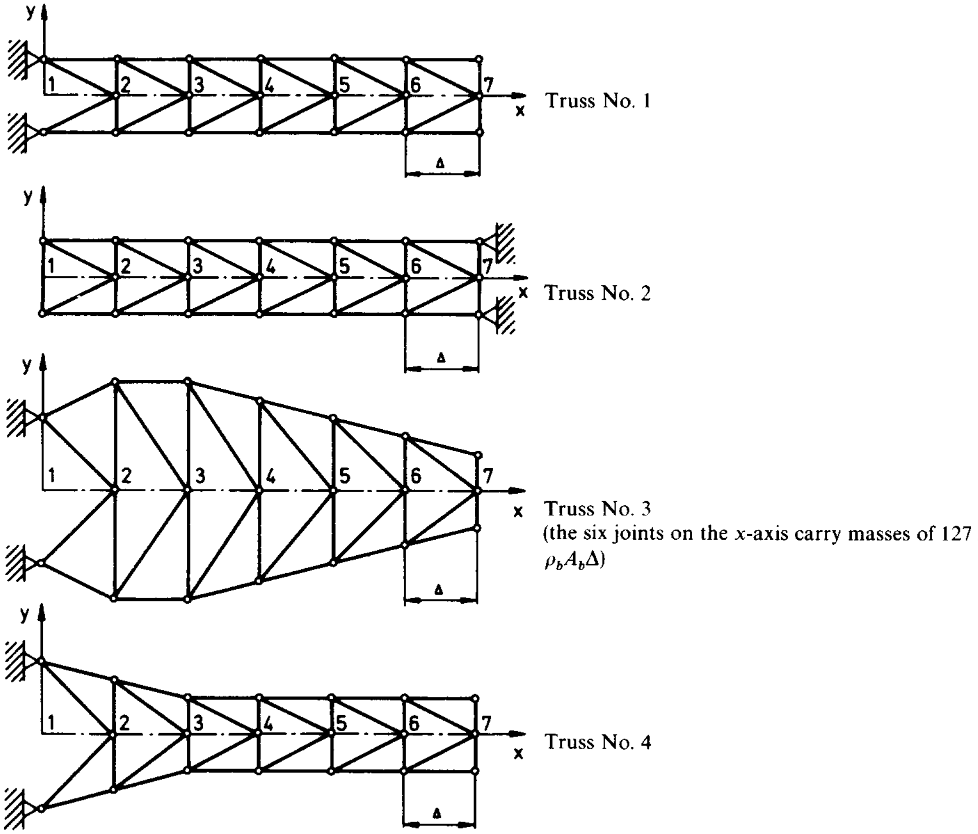
The trusses shown in Fig. 5 are composed of prismatic bars with the cross-sectional area A_b, Young's modulus E_b, the mass density ρ_b, and the area moment of inertia I_b. The beam functions at the middle of each of the seven different sections appearing in these plane trusses are given in Table 2.

Timoshenko beams have the constitutive equations

$$\frac{dv}{dx} = \frac{1}{EI}M \tag{5.3}$$

$$\frac{dy}{dx} - v = \frac{\mu}{GA}T. \tag{5.4}$$

For the section of a plane truss shown in Fig. 4 the values of the flexibility functions $1/EI$ and μ/GA may be determined by a method analogous to equations (5.1) and (5.2). This



$\sqrt{(\lambda_1)} \cdot \sqrt{(\Delta^2 \rho_b / E_b)} \cdot 10^4$	True value (Appendix A)	Bernoulli–Euler theory	Timoshenko theory	Present beam theory
Truss No. 1	301	290 (−4%)	263 (−13%)	307 (+2,0%)
Truss No. 2	230	290 (+26%)	263 (+14%)	233 (+1,3%)
Truss No. 3	91	109 (+20%)	87 (−4%)	89 (−2,2%)
Truss No. 4	356	307 (−14%)	277 (−22%)	366 (+2,8%)

FIG. 5. Smallest natural frequencies of four plane trusses. Parentheses contain relative errors in per cent.

TABLE 2. BEAM FUNCTIONS

	Truss No. 1	Truss No. 3: 1	2	3	4	5	6	7
$a_{11} \cdot E_b A_b \Delta^2 \cdot 10^3$	2000	551	222	428	661	1172	2687	
$a_{12} \cdot E_b A_b \Delta \cdot 10^3$	−1000	446	−111	−599	−784	−1183	−2337	
$a_{21} \cdot E_b A_b \Delta \cdot 10^3$	−1000	446	−111	−599	−784	−1183	−2337	
$a_{22} \cdot E_b A_b \cdot 10^3$	3545	1581	2108	2419	2431	2679	3651	

leads to the values of the functions a_{11} and a_{22} given in Table 2. The Bernoulli–Euler theory neglects the function μ/GA.

Now, considering the four trusses of Figure 5 as beams, we compute the natural frequencies of these structures by applying each of the three beam theories. The first natural frequency is given in the table of Fig. 5 for each case.

For comparison we use an exact iterative method that also takes into account the transverse vibrations of the bars (Appendix A). The slenderness of the bars is determined by the parameter $I_b/(A_b\Delta^2) = 1{\cdot}30 \,.\, 10^{-4}$, where Δ is the length of one section. The effect of these transverse vibrations is rather unimportant for all four trusses considered, since the first natural frequency of these structures is much smaller than the smallest natural frequency of any of the bars. However, if the six concentrated masses are removed from truss No. 3, the beam theories result in a first natural frequency, which is much higher than the true value. This is due to the fact that the first natural frequency of a truss must always be less than the smallest natural frequency of any of the bars.

In the table of Fig. 5 the results of the beam theories for the trusses are compared with the exact values. The beam theory presented here leads to the best result in all four cases. Truss No. 1 and truss No. 2 are actually the same structure clamped at different ends, and we note that the Bernoulli–Euler and the Timoshenko beam theories lead to the same results for the two structures, whereas the present beam theory describes the difference. Although the Timoshenko theory usually leads to better results than the Bernoulli–Euler theory, surprisingly this is not the case for two of the four examples. The reason is that there is one important property of the trusses that cannot be described by either of these two theories. This fact can be illustrated by the following example: a transverse load at cross-section 2 of truss No. 4 results in a deflection at the end of the "beam" in the opposite direction to the load. Neither the Timoshenko theory nor the Bernoulli–Euler theory could lead to this result. However, the present beam theory accounts for such behaviour.

Trusses are just one example of beam-like structures that can be treated by this generalized beam theory. It is presumably also applicable to a number of other structures such as beam-like frames and shells. The application to a class of axisymmetrical thin shells is currently being investigated by the author.

Acknowledgements—I am indebted to Professor Ph.D. Frithiof I. Niordson for his inspiring suggestions in the course of this work. I also wish to express thanks to lic. techn. Preben Terndrup Pedersen and to civ. ing. Pauli Pedersen for many valuable comments.

REFERENCES

[1] M. Bresse, *Cours de Mecanique Appliquée*. Première Partie. Mallet-Bachelier (1859).
[2] S. P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Phil. Mag*. **41**, 744–746 (1921).
[3] R. A. Anderson, Flexural vibrations in uniform beams according to the Timoshenko theory. *J. appl. Mech*. **20**, 504–510 (1953).
[4] T. C. Huang, Effect of rotatory inertia and shear on the vibration of beams treated by the approximate methods of Ritz and Galerkin. *Proceedings of the Third U.S. National Congress of Applied Mechanics*, pp. 189–194 (1958).
[5] G. Herrmann, Forced motions of Timoshenko beams. *J. appl. Mech*. **22**, 53–56 (1955).
[6] E. Kamke, *Differentialgleichungen* I, 4. auflage. Akademische Verlagsgesellschaft (1962).
[7] L. Collatz, *Eigenwertaufgaben mit Technischen Anwendungen*, 2. auflage. Akademische Verlagsgesellschaft (1963).

APPENDIX A

Vibrations of plane trusses

When a plane truss vibrates freely, the d'Alembert loads result in transverse deflections of the bars. Taking into consideration the effect of these deflections, we determine the smallest natural frequency of the truss by means of the following iterative procedure.

Step (a). The struss is assumed to vibrate freely with a certain frequency Ω, and since we neglect damping, the bars have to vibrate in the same frequency and phase as the whole structure. Figure A1 shows one of the bars in forced vibration with frequency Ω. The

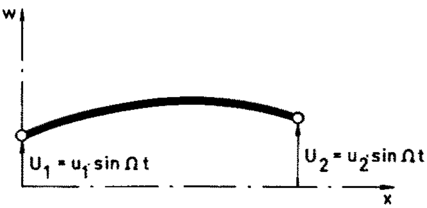


FIG. A1. One bar in vibration.

differential equation for the deflection of a bar:

$$(EIw'')'' = \Omega^2 \rho A w \tag{A1}$$

has the general solution

$$w(x) = C_1 \cosh(rx) + C_2 \sinh(rx) + C_3 \cos(rx) + C_4 \sin(rx) \tag{A2}$$

where the constant r is given by the expression

$$r^4 = \frac{\rho A \Omega^2}{EI}. \tag{A3}$$

The four integration constants are determined from the boundary conditions:

$$w(0) = u_1, w''(0) = 0, w(l) = u_2, w''(l) = 0 \tag{A4}$$

and having done this for all bars of the truss, we determine Rayleigh's quotient from the maximum strain energy and the maximum kinetic energy of the whole structure. We thus have a new estimate of the frequency Ω, and step (a) is repeated until Ω is stable.

Step (b). The loads at the joints equivalent to the maximum d'Alembert loads on the whole structure are calculated, and again making use of the stiffness method, we obtain the joint displacements due to these loads. We then return to Step (a).

The iteration is stopped when the value of Ω passing Step (b) has become stable.

Even though the iteration converges, it does not always converge towards the smallest frequency unless special measures are taken. In this programme we make use of the fact that the first natural frequency of the truss is always smaller than the first natural frequency of any of the bars.

(*Received* 13 *June* 1970; *revised* 28 *September* 1970)

Абстракт—Целью настоящей работы является представление обобщенной теории балки, с помощью которой можно соответственно решить некоторое число балкоподобных конструкций в смысле балок.

Выводятся уравнения движения, заключающие эффект инерции вращения, Рассматривая свободные гпрмонические колебания, в результате решается задача на собственные значения, которая состоит из двух совместных дифференциальных уравнений второго порядка и четырых граничесских условий, с помощью общепринятым способом последовательных приближений.

Представляется достоинства этой теории на нескольких простых примерах. Решаются свободные частоты некоторых балкоподобных ферм. Результаты сравниваются с результатами теории балок Бернулли-Эйлера и Тимошенки.